GCE: Analysis, measure theory, Lebesgue integration

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Exercise 1:
Give an example of $f_n, f \in L^1(\mathbb{R})$ such that $f_n \to f$ uniformly, but $\|f_n\|_1$ does not converge to $\|f\|_1$.

Exercise 2:
Show that for all $\epsilon > 0$ and all $f \in L^1(\mathbb{R})$, $\exists n \in \mathbb{N}$ such that $\|f - f_n\|_1 < \epsilon$ for some f_n with $|f_n| \leq n$ and $f_n = 0$ on $\mathbb{R} \setminus [-n, n]$.

Exercise 3:
Let $(X, \mathcal{A}, \mu)$ be a measure space.
(i). If f is in $L^1(X) \cap L^\infty(X)$, show that $|f|^p \in L^1(X)$ for all p in $(1, \infty)$.
(ii). If f is in $L^1(X) \cap L^\infty(X)$, show that

$$\lim_{p \to \infty} (\int |f|^p)^{1/p} = \|f\|_\infty.$$

(iii). Set $A = \{x \in X : |f(x)| > 0\}$. If f is in $L^\infty(X)$, $\mu(A) < \infty$, and $\mu(A) \neq 1$, find

$$\lim_{p \to 0^+} (\int |f|^p)^{1/p}.$$

Hint: It may be helpful to cover separately the case where $\|f\|_\infty = 1$.
(iv). We now assume that the set A defined in (iii). satisfies $\mu(A) = 1$, that f is in $L^\infty(X)$, and $\ln |f|$ is in $L^1(X)$, find

$$\lim_{p \to 0^+} (\int |f|^p)^{1/p}.$$